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# Green's functions of an interfacial crack between two dissimilar piezoelectric media

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**Abstract**

Based on the permeable crack model, the Green's function of an interfacial crack between two dissimilar piezoelectric media is first presented by means of the Stroh formalism. Then, the electric field inside the crack and the fundamental solution of field intensity factor near the crack tips are obtained in an explicit closed-form. As special examples, several Green's functions are given for the cases of a crack in a homogeneous piezoelectric material, an interfacial crack between two dissimilar purely-elastic media, and a bimaterial of piezoelectric materials, respectively. It is shown in the general case that all the field variables near the crack-tips are singular and oscillatory, and such is the case for the electric field inside the crack when approaching to the crack-tips from on the crack faces. In addition, the relation between the field intensity factor vector of an impermeable crack and that of a permeable crack is established in the general cases. © 2001 Elsevier Science Ltd. All rights reserved.

*Keywords:* Piezoelectric; Interface crack; Green's function

## 1. Introduction

With increasingly-wide application of piezoelectric materials in engineering, the crack problem in an infinite piezoelectric medium has received considerable interest in the last years. However, to the authors' knowledge, there is still no solution to the case of a finite-size piezoelectric material with cracks, though it is often met in normal applications. In this case, numerical approaches may become only a reasonable alternative. The boundary element method (BEM), with dual nature of analytical method and numerical method, has been considered to be a good alternative for treating the complicated problems in piezoelectric media. The analytical nature of BEM is reflected in its fundamental solution, i.e., Green's function, which is the heart of this method. Thus, it is important to study the Green's function of piezoelectric solids. Many efforts, indeed, have been made in this direction, which can be found in the recent works by Wang (1992), Benveniste (1992), Chen (1993), Lee and Jiang (1994), Dunn (1994), Dunn and Wienecke (1996), Ding and Chen (1997), Akamatsu and Tanuma (1997), Ding et al. (1997), Gao and Fan (1998a), and Dunn and Wienecke (1999). However, it should be noted that the above cited studies were focused on a piezoelectric

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material without hole. In fact, it is more important to study the Green's functions of an infinite (2D) piezoelectric medium containing a hole or crack. Since the presence of the hole or crack has already been taken into account in the special Green's functions which satisfy the boundary conditions on the hole or crack surface (zone of the high stress concentration), it is not necessary to consider any more these boundary conditions in BEM analyses, and therefore more accurate results can be obtained. More recently, Liu et al. (1997), and Lu and Williams (1998) presented the Green's functions of an infinite two-dimensional (2D) piezoelectric material with an impermeable elliptic hole, respectively. Gao and Fan (1998b) resolved this problem according to the exact electric boundary condition at the rim of hole and gave the Green's function for a permeable crack in a generally anisotropic piezoelectric medium. In their another paper, Gao and Fan (1999) addressed a collinear permeable crack problem in a transversely isotropic piezoelectric solid subject to the most general loading. As a special case, they presented the Green's function of a crack. However, Gao and Fan's work (1998b, 1999) were for the cases where a crack is in a homogeneous piezoelectric material.

In the present work, Green's functions are presented for the generalized 2D problems of an interfacial crack between two dissimilar piezoelectric media by use of the extended Stroh formalism. The analysis is still based on the permeable crack model, i.e., the crack is treated as a permeable slit and thus both the normal components of electric displacement and the tangential component of electric field are assumed to be continuous across the slit faces (Parton, 1976; Gao and Yue, 1998c; Gao and Fan, 1999; Gao and Wang, 1999; Han and Chen, 1999; Wang and Han, 1999; Gao and Wang, 2000). The whole content consists of six sections. Following this brief introduction, Section 2 outlines the Stroh formalism to be need in this paper, and then the fundamental solutions of complex potentials and the field intensity factors are presented in Sections 3 and 4, respectively. In Section 5 given are the solutions of several special examples, including those of a crack in a homogeneous medium, an interfacial crack between two dissimilar purely-elastic anisotropic materials, and a bimaterial of piezoelectric materials without crack. Finally, Section 6 concludes the present work.

## 2. Stroh formalism

Consider a piezoelectric solid in a Cartesian system $x_j$ $(j = 1, 2, 3)$. Assuming that the displacement $u_j$ and electric potential $\varphi$ of the solid are dependent on $x_1$ and $x_2$ only, then the general solution for the generalized 2D problem can be expressed as (Suo et al. 1992):

$$\begin{aligned} \mathbf{u} &= \mathbf{A}\mathbf{f}(z) + \overline{\mathbf{A}}\,\overline{\mathbf{f}(z)} \\ \phi &= \mathbf{B}\mathbf{f}(z) + \overline{\mathbf{B}}\,\overline{\mathbf{f}(z)} \end{aligned} \tag{1}$$

with

$$\mathbf{u} = [u_1, u_2, u_3, \varphi]^{\mathrm{T}}, \qquad \phi = [\phi_1, \phi_2, \phi_3, \phi_4]^{\mathrm{T}}$$

$$\mathbf{f}(z) = [f_1(z_1), f_2(z_2), f_3(z_3), f_4(z_4)]^{\mathrm{T}}, \quad z_\alpha = x_1 + p_\alpha x_2 \ \ (\alpha = 1\text{–}4)$$

In the above equations, $\mathbf{u}$ and $\phi$ denote the generalized displacement function vector and stress function vector, respectively; the overbar stands for the conjugate of a complex number; $\mathbf{A}$ and $\mathbf{B}$ are two $4 \times 4$ matrices which can be determined from the material constants; $f_\alpha(z_\alpha)$ are complex potentials to be found; the superscript T represents the transpose; $p_\alpha$ $(\alpha = 1\text{–}4)$ are the complex eigenvalues with positive imaginary parts; In this paper we assume that $p_\alpha$ are distinct.

Once $\mathbf{f}(z)$ is obtained according to the given boundary conditions, the stress $\sigma_{jl}$, electric displacement $D_l$ and electric field $E_l$ can be given, respectively, by

$$\sigma_{j1} = -\phi_{j,2}, \qquad \sigma_{j2} = \phi_{j,1} \quad (j = 1, 2, 3) \tag{2}$$

$$D_1 = -\phi_{4,2}, \quad D_2 = \phi_{4,1}, \qquad E_1 = -u_{4,1}, \quad E_2 = -u_{4,2} \tag{3}$$

where a comma indicates partial differentiation.

In the present work, we will adopt the one-complex-variable approach introduced by Suo (1990), i.e., the arguments of each component function of $\mathbf{f}(z)$ are written as $z = x_1 + \mu x_2$ without referring to the associated eigenvalues $\mu_\alpha$. After the solution of $\mathbf{f}(z)$ is obtained, one should substitute $z_1$, $z_2$, $z_3$ or $z_4$ for each component function of $\mathbf{f}(z)$ to calculate field quantities.

## 3. The Greens functions

Consider two dissimilar piezoelectric solids, one located in the upper half space $S_1$, and the other in the lower half space $S_2$. An interface cracks $L_c$ lies on the real axis $x_1$ along $[-a, a]$, and the uncracked part in the $x_1$-axis is denoted $L_b$. Moreover, it is assumed that the crack is a traction-free, but permeable slit filled with air or vacuum, while the upper space is subjected a line force $(q_{10}, q_{20}, q_{30})$ associated with a line charge $q_{40}$ at an arbitrary point $z_0 = x_{10} + \mathrm{i}x_{20}$, as shown in Fig. 1.
On the crack faces, the boundary conditions can be written as

$$\sigma_{2j}^+ = \sigma_{2j}^- = 0 \quad (j = 1, 2, 3) \text{ on } L_c \tag{4}$$

$$D_2^+ = D_2^-, \qquad E_1^+ = E_1^- \quad \text{on } L_c \tag{5}$$

On the bonded part, the continuous condition requires

$$\sigma_{2j}^+ = \sigma_{2j}^-, \qquad u_j^+ = u_j^-, \quad (j = 1, 2, 3) \text{ on } L_b \tag{6}$$

$$D_2^+ = D_2^-, \qquad E_1^+ = E_1^- \quad \text{on } L_b \tag{7}$$

For convenience of the later use, Eqs. (4)–(7) can be rearranged into

$$\sigma_{2j}^+ = \sigma_{2j}^-, \qquad D_2^+ = D_2^-, \quad -\infty < x_1 < +\infty \tag{8}$$

$$E_1^+ = E_1^-, \quad -\infty < x_1 < +\infty \tag{9}$$

$$u_{j,1}^+ = u_{j,1}^-, \quad \text{on } L_b \tag{10}$$

$$\sigma_{2j}^+ = 0, \quad \text{on } L_c \tag{11}$$

For the present problem, we have

$$\mathbf{F}_l(z) = \delta_{l1}^0 \mathbf{G}_{10}(z) + \mathbf{F}_{l0}(z) \quad (l = 1, 2) \tag{12}$$

where $\mathbf{F}(z) = \mathrm{d}\mathbf{f}(z)/\mathrm{d}z$; $\delta_{l1}^0$ stands for the Kronecker notation; $\mathbf{F}_{l0}(z)$ is a holomorphic function vector in $S_1$ $(l = 1)$ or $S_2$ $(l = 2)$, and $\mathbf{F}_{l0}(\infty) = \mathbf{0}$; while $\mathbf{G}_{10}(z)$ is a singular function vector at the point $z_{\alpha 0} = x_{10} + p_\alpha x_{20}$ such that

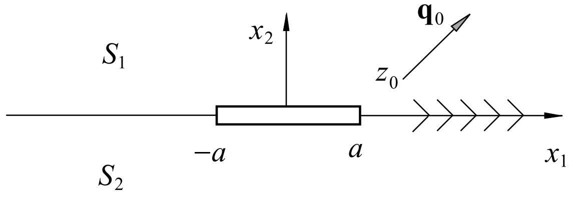


Fig. 1. An interfacial crack subjected to an arbitrary line load.

$$\mathbf{G}_{10}(z) = \left\langle\left\langle \frac{1}{z - z_{\alpha 0}} \right\rangle\right\rangle \mathbf{q} \tag{13}$$

In Eq. (13), the angular bracket $\langle\langle\ \rangle\rangle$ indicates the diagonal matrix in which each component is varied according to the Greek index $\alpha$, and $\mathbf{q}$ is a constant vector, which is determined by (Gao and Fan, 1998b)

$$\mathbf{q} = \frac{1}{2\pi i}\mathbf{A}^{\mathrm{T}}\mathbf{q}_0, \qquad \mathbf{q}_0 = (q_{10}, q_{20}, q_{30}, q_{40})^{\mathrm{T}}$$

Using Eq. (8), the continuity of $\sigma_{2j}$ and $D_2$ on the whole $x_1$ axis implies

$$\mathbf{B}_1\mathbf{F}_{10}(x_1) + \overline{\mathbf{B}_1}\,\overline{\mathbf{F}_{10}(x_1)} + \mathbf{B}_1\mathbf{G}_{10}(x_1) + \overline{\mathbf{B}_1}\,\overline{\mathbf{G}_{10}(x_1)} = \mathbf{B}_2\mathbf{F}_{20}(x_1) + \overline{\mathbf{B}_2}\,\overline{\mathbf{F}_{20}(x_1)} \quad -\infty < x_1 < +\infty \tag{14}$$

Define an analytical function $\mathbf{h}(z)$ as:

$$\mathbf{h}(z) = \begin{cases} \mathbf{B}_1\mathbf{F}_{10}(z) - \overline{\mathbf{B}_2}\,\overline{\mathbf{F}_{20}}(z) + \overline{\mathbf{B}_1}\,\overline{\mathbf{G}_{10}}(z), & z \in S_1 \\ \mathbf{B}_2\mathbf{F}_{20}(z) - \overline{\mathbf{B}_1}\,\overline{\mathbf{F}_{10}}(z) - \mathbf{B}_1\mathbf{G}_{10}(\mathrm{z}), & z \in S_2 \end{cases} \tag{15}$$

Then, Eq. (14) can be reduced to

$$\mathbf{h}^+(x_1) - \mathbf{h}^-(x_1) = \mathbf{0} \quad -\infty < x_1 < +\infty \tag{16}$$

The solution of Eq. (15) is given (Muskhelishvili, 1975) by $\mathbf{h}(z) = \mathbf{0}$, and thus Eq. (15) leads to

$$\mathbf{B}_1\mathbf{F}_{10}(z) - \overline{\mathbf{B}_2}\,\overline{\mathbf{F}_{20}}(z) + \overline{\mathbf{B}_1}\,\overline{\mathbf{G}_{10}}(z) = \mathbf{0}, \quad z \in S_1 \tag{17}$$

$$\mathbf{B}_2\mathbf{F}_{20}(z) - \overline{\mathbf{B}_1}\,\overline{\mathbf{F}_{10}}(z) - \mathbf{B}_1\mathbf{G}_{10}(\mathrm{z}) = \mathbf{0}, \quad z \in S_2 \tag{18}$$

Further, introduce two auxiliary functions:

$$\Delta\mathbf{U}(x_1) = \mathbf{u}_{1,1}(x_1) - \mathbf{u}_{2,1}(x_1) = \left[\mathbf{A}_1\mathbf{F}_1(x_1) + \overline{\mathbf{A}_1}\,\overline{\mathbf{F}_1(x_1)}\right] - \left[\mathbf{A}_2\mathbf{F}_2(x_1) + \overline{\mathbf{A}_2}\,\overline{\mathbf{F}_2(x_1)}\right] \tag{19}$$

$$\mathbf{T}(x_1) = \mathbf{B}_1\mathbf{F}_1(x_1) + \overline{\mathbf{B}_1}\,\overline{\mathbf{F}_1(x_1)} \tag{20}$$

Then, using Eqs. (17) and (18), Eq. (19) reduces to

$$\mathrm{i}\Delta\mathbf{U}(x_1) = \mathbf{H}\left[\mathbf{B}_1\mathbf{F}_{10} + \mathbf{H}^{-1}\left(\overline{\mathbf{Y}}_2 - \overline{\mathbf{Y}}_1\right)\overline{\mathbf{B}_1}\,\overline{\mathbf{G}_{10}} - \mathbf{H}^{-1}\overline{\mathbf{H}}\mathbf{B}_2\mathbf{F}_{20} + \mathbf{H}^{-1}\left(\mathbf{Y}_1 + \overline{\mathbf{Y}}_1\right)\mathbf{B}_1\mathbf{G}_{10}\right] \tag{21}$$

where

$$\mathbf{Y}_1 = \mathrm{i}\mathbf{A}_1\mathbf{B}_1^{-1}, \qquad \mathbf{Y}_2 = \mathrm{i}\mathbf{A}_2\mathbf{B}_2^{-\mathbf{1}} \qquad \mathbf{H} = \mathbf{Y}_1 + \overline{\mathbf{Y}}_2$$

Suo et al. (1992) have shown that when $p_\alpha$ ($\alpha = 1$–$4$) are distinct, the matrix $\mathbf{H}$, which has the same properties as $\mathbf{Y}_1$ and $\mathbf{Y}_2$, is Hermitian and non-singular.

Moreover, Eq. (21) can be rewritten as

$$\mathrm{i}\Delta\mathbf{U}(x_1) = \mathbf{H}[\mathbf{K}^+(x_1) - \mathbf{K}^-(x_1)] \tag{22}$$

where $\mathbf{K}(z)$ is a newly introduced vector defined as

$$\mathbf{K}(z) = \begin{cases} \mathbf{B}_1\mathbf{F}_{10}(z) + \mathbf{H}^{-1}\left(\overline{\mathbf{Y}}_2 - \overline{\mathbf{Y}}_1\right)\overline{\mathbf{B}_1}\,\overline{\mathbf{G}_{10}}(z) & z \in S_1 \\ \mathbf{H}^{-1}\overline{\mathbf{H}}\mathbf{B}_2\mathbf{F}_{20}(z) - \mathbf{H}^{-1}\left(\mathbf{Y}_1 + \overline{\mathbf{Y}}_1\right)\mathbf{B}_1\mathbf{G}_{10}(z) & z \in S_2 \end{cases} \tag{23}$$

Noting from Eqs. (9) and (10) that $\Delta\mathbf{U}(x_1) = 0$ on $L_{\mathrm{b}}$, and therefore Eq. (22) shows that except on $L_{\mathrm{c}}$, $\mathbf{K}(z)$ is analytic in the entire $z$-plane up to at infinity such that

$$\mathbf{K}(\infty) = \mathbf{0} \tag{24}$$

In addition, the single-valued conditions of displacement and electric potentials means

$$\left[u_j^+(+a) - u_j^+(-a)\right] + \left[u_j^-(-a) - u_j^-(+a)\right] = 0, \quad (j = 1\text{–}4) \tag{25}$$

Eq. (25) can be rewritten as

$$\int_{-a}^{+a} u_{j,1}^+ \, \mathrm{d}x_1 + \int_{+a}^{-a} u_{j,1}^- \, \mathrm{d}x_1 = 0 \tag{26}$$

namely

$$\int_{-a}^{+a} \left(u_{j,1}^+ - u_{j,1}^-\right) \mathrm{d}x_1 = 0 \tag{27}$$

Using Eq. (19), Eq. (27) can be expressed, in the form of vector, as

$$\int_{-a}^{+a} \Delta\mathbf{U}(x_1)\, \mathrm{d}x_1 = \mathbf{0} \tag{28}$$

Inserting Eq. (22) into Eq. (28) leads to

$$\oint_{\Gamma} \mathbf{K}(z)\, \mathrm{d}z = \mathbf{0} \tag{29}$$

where $\Gamma$ stands for a clockwise closed-contour closing in on the crack (in this case, $z \to x_1$).

On the other hand, using the continuous condition of $E_1$, i.e. Eq. (9), one can obtain from Eq. (22) that

$$\mathbf{H}_4[\mathbf{K}^+(x_1) - \mathbf{K}^-(x_1)] = \mathbf{0}, \quad -\infty < x_1 < \infty \tag{30}$$

where

$$\mathbf{H}_4 = (H_{41}, H_{42}, H_{43}, H_{44})$$

Noting Eq. (24), the solution of Eq. (30) is

$$\mathbf{H}_4\mathbf{K}(z) = \mathbf{0} \tag{31}$$

Similarly, by using Eqs. (17) and (18), one can obtain from Eq. (20) that

$$\mathbf{T}(x_1) = \mathbf{K}^+(x_1) + \overline{\mathbf{H}}^{-1}\mathbf{H}\mathbf{K}^-(x_1) + \overline{\mathbf{H}}^{-1}\left(\mathbf{Y}_1 + \overline{\mathbf{Y}_1}\right)\mathbf{B}_1\mathbf{G}_{10}(x_1) + \mathbf{H}^{-1}\left(\mathbf{Y}_1 + \overline{\mathbf{Y}_1}\right)\overline{\mathbf{B}_1}\,\overline{\mathbf{G}_{10}(x_1)},$$
$$-\infty < x_1 < +\infty \tag{32}$$

On the crack faces, one has from Eq. (11) that $\mathbf{T}(x_1) = \mathbf{i}_4 D_2(x_1)$, where $\mathbf{i}_4 = (0,0,0,1)^{\mathrm{T}}$ and $D_2(x_1)$ is an unknown function which indicates the boundary value of $D_2(z)$ on the crack faces. Hence, Eq. (32) reduces

$$\mathbf{K}^+(x_1) + \overline{\mathbf{H}}^{-1}\mathbf{H}\mathbf{K}^-(x_1) = \mathbf{i}_4 D_2(x_1) - \overline{\mathbf{H}}^{-1}\left(\mathbf{Y}_1 + \overline{\mathbf{Y}_1}\right)\mathbf{B}_1\mathbf{G}_{10}(x_1) - \mathbf{H}^{-1}\left(\mathbf{Y}_1 + \overline{\mathbf{Y}_1}\right)\overline{\mathbf{B}_1}\,\overline{\mathbf{G}_{10}(x_1)},$$
$$x_1 \in L_c \tag{33}$$

Letting $\mathbf{Q}$ be the eigenvector matrix of $\overline{\mathbf{H}}^{-1}\mathbf{H}$, one has

$$\mathbf{Q}^{-1}\overline{\mathbf{H}}^{-1}\mathbf{H}\mathbf{Q} = \Lambda, \qquad \Lambda = \langle\langle -\mathrm{e}^{-2\pi\mathrm{i}\delta_\alpha}\rangle\rangle = \langle\langle \mathrm{e}^{2\pi\varepsilon_\alpha}\rangle\rangle, \qquad \delta_\alpha = -\tfrac{1}{2} + \mathrm{i}\varepsilon_\alpha \tag{34}$$

where $\varepsilon_\alpha$ is given by (Suo et al., 1992)

$$\left\|\mathbf{H} - \mathrm{e}^{2\pi\varepsilon_\alpha}\overline{\mathbf{H}}\right\| = 0 \tag{35}$$

Define a new vector $\mathbf{R}(z)$ as

$$\mathbf{R}(z) = \mathbf{Q}^{-1}\mathbf{K}(z) \tag{36}$$

Then, Eq. (33) becomes

$$\mathbf{R}^{+}(x_1) + \langle\langle -\mathrm{e}^{-2\pi i\delta_\alpha}\rangle\rangle\mathbf{R}^{-}(x_1) = \mathbf{M}^{(0)}D_2(x_1) - \mathbf{M}^{(1)}\mathbf{G}_{10}(x_1) - \mathbf{M}^{(2)}\overline{\mathbf{G}_{10}(x_1)}, \quad x_1 \in L_c \tag{37}$$

where

$$\mathbf{M}^{(0)} = \mathbf{Q}^{-1}\mathbf{i}_4, \qquad \mathbf{M}^{(1)} = \mathbf{Q}^{-1}\overline{\mathbf{H}}^{-1}(\mathbf{Y}_1 + \overline{\mathbf{Y}_1})\mathbf{B}_1, \qquad \mathbf{M}^{(2)} = \mathbf{Q}^{-1}\mathbf{H}^{-1}(\mathbf{Y}_1 + \overline{\mathbf{Y}_1})\overline{\mathbf{B}_1}$$

Eq. (37) can be expanded into

$$R_\alpha^{+}(x_1) - g_\alpha R_\alpha^{-}(x_1) = f_\alpha(x_1), \quad (\alpha = 1\text{–}4) \tag{38}$$

where $g_\alpha$ and $f_\alpha(x_1)$ are given in the Appendix A.

The solution of Eq. (38) is derived in detail in the Appendix A. The result is

$$\begin{aligned} R_\alpha(z) = {} & \frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}}\left[M_\alpha^{(0)}D_2(z) - \sum_{j=1}^{4}M_{\alpha j}^{(1)}q_j(z-z_{j0})^{-1} - \sum_{j=1}^{4}M_{\alpha j}^{(2)}\overline{q_j}(z-\overline{z_{j0}})^{-1}\right] \\ & + \frac{X_\alpha(z)}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}}\left[\sum_{j=1}^{4}M_{\alpha j}^{(1)}q_j\left(1+\frac{X_\alpha^{-1}(z_{j0})}{z-z_{j0}}\right) + \sum_{j=1}^{4}M_{\alpha j}^{(2)}\overline{q_j}\left(1+\frac{X_\alpha^{-1}(\overline{z_{j0}})}{z-\overline{z_{j0}}}\right)\right] \\ & + \frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}}X_\alpha(z)\left[c_\alpha^{(1)}z + c_\alpha^{(0)}\right] \end{aligned} \tag{39}$$

where $X_\alpha(z)$ is given by Eq. (A.10), and $c_\alpha^{(1)}$ and $c_\alpha^{(0)}$ are constants to be determined. Using Eqs. (24) and (29), one has from Eq. (36) that

$$\mathbf{R}(\infty) = \mathbf{0}, \qquad \oint_\Gamma \mathbf{R}(z)\,\mathrm{d}z = \mathbf{0} \tag{40}$$

Eq. (40) gives

$$R_\alpha(\infty) = 0, \tag{41a}$$

$$\oint_\Gamma R_\alpha(z)\,\mathrm{d}z = 0 \tag{41b}$$

Taking the limit $z \to \infty$ in Eq. (39), and then using Eq. (41a) yields

$$c_\alpha^{(1)} = 0 \tag{42}$$

Substituting Eq. (39) into Eq. (41b), and then using the Gauss' law:

$$\oint_\Gamma D_2(z)\,\mathrm{d}z = 0$$

and the residue theorem:

$$\oint_\Gamma \frac{1}{z-z_{j0}}\,\mathrm{d}z = 0, \qquad \oint_\Gamma \frac{1}{z-\bar{z}_{j0}}\,\mathrm{d}z = 0$$

$$\oint_\Gamma X_\alpha(z)\,\mathrm{d}z = -1, \qquad \oint_\Gamma zX_\alpha(z)\,\mathrm{d}z = 0$$

$$\oint_{\Gamma} \frac{X_\alpha(z)X_\alpha^{-1}(z_{j0})}{z-z_{j0}}\,\mathrm{d}z = 1, \qquad \oint_{\Gamma} \frac{X_\alpha(z)X_\alpha^{-1}(\bar{z}_{j0})}{z-\bar{z}_{j0}}\,\mathrm{d}z = 1$$

we have

$$c_\alpha^{(0)} = 0 \tag{43}$$

Inserting Eqs. (42) and (43) into Eq. (39) and then rewriting $R_\alpha(z)$ in the matrix form, we have

$$\begin{aligned}\mathbf{R}(z) = {}& \left\langle\left\langle \frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}} \right\rangle\right\rangle \left[\mathbf{M}^{(0)}D_2(z) - \mathbf{M}^{(1)}\mathbf{G}_{10}(z) - \mathbf{M}^{(2)}\overline{\mathbf{G}_{10}}(z)\right] \\ & + \left\langle\left\langle \frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}} \right\rangle\right\rangle \langle\langle X_\alpha(z)\rangle\rangle \left[\sum_{j=1}^{4}\left\langle \mathbf{W}_j^{(1)} + \mathbf{I}\right\rangle \mathbf{M}^{(1)}\mathbf{I}_j\mathbf{q} + \sum_{j=1}^{4}\left\langle \mathbf{W}_j^{(2)} + \mathbf{I}\right\rangle \mathbf{M}^{(2)}\mathbf{I}_j\bar{\mathbf{q}}\right]\end{aligned} \tag{44}$$

where

$$\mathbf{W}_j^{(1)} = \mathrm{diag}\left[\frac{X_1^{-1}(z_{j0})}{z-z_{j0}}, \frac{X_2^{-1}(z_{j0})}{z-z_{j0}}, \frac{X_3^{-1}(z_{j0})}{z-z_{j0}}, \frac{X_4^{-1}(z_{j0})}{z-z_{j0}}\right]$$

$$\mathbf{W}_j^{(2)} = \mathrm{diag}\left[\frac{X_1^{-1}(\overline{z_{j0}})}{z-\overline{z_{j0}}}, \frac{X_2^{-1}(\overline{z_{j0}})}{z-\overline{z_{j0}}}, \frac{X_3^{-1}(\overline{z_{j0}})}{z-\overline{z_{j0}}}, \frac{X_4^{-1}(\overline{z_{j0}})}{z-\overline{z_{j0}}}\right]$$

$$\mathbf{I} = \mathrm{diag}[1,1,1,1], \qquad \mathbf{I}_1 = \mathrm{diag}[1,0,0,0], \qquad \mathbf{I}_2 = \mathrm{diag}[0,1,0,0], \qquad \mathbf{I}_3 = \mathrm{diag}[0,0,1,0],$$
$$\mathbf{I}_4 = \mathrm{diag}[0,0,0,1]$$

From Eq. (31), one has

$$\mathbf{H}_4\mathbf{Q}\mathbf{R}(z) = \mathbf{0} \tag{45}$$

Substituting Eq. (44) into Eq. (45), one can obtain the expression of $D_2(z)$. The result is

$$\begin{aligned}D_2(z) = {}& \frac{1}{c_D}\mathbf{H}_4\mathbf{Q}\left\langle\left\langle \frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}} \right\rangle\right\rangle \left[\mathbf{M}^{(1)}\mathbf{G}_{10}(z) + \mathbf{M}^{(2)}\overline{\mathbf{G}_{10}}(z)\right] \\ & - \frac{1}{c_D}\mathbf{H}_4\mathbf{Q}\left\langle\left\langle \frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}} \right\rangle\right\rangle \langle\langle X_\alpha(z)\rangle\rangle \left[\sum_{j=1}^{4}\left\langle \mathbf{W}_j^{(1)} + \mathbf{I}\right\rangle \mathbf{M}^{(1)}\mathbf{I}_j\mathbf{q} + \sum_{j=1}^{4}\left\langle \mathbf{W}_j^{(2)} + \mathbf{I}\right\rangle \mathbf{M}^{(2)}\mathbf{I}_j\bar{\mathbf{q}}\right]\end{aligned} \tag{46}$$

where

$$c_D = \mathbf{H}_4\mathbf{Q}\left\langle\left\langle \frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}} \right\rangle\right\rangle \mathbf{Q}^{-1}\mathbf{i}_4$$

Substituting Eq. (46) into Eq. (44), one can obtain the complete expression of $\mathbf{K}(z)$, and then $\mathbf{F}_{j0}(z)$ can be determined by using Eq. (23). With $\mathbf{F}_{j0}(z)$, all the field variables in the media can be determined by using Eqs. (1)–(3).

## 4. The fundamental solutions of intensity factors

For a permeable crack, noting from Eq. (22) that $\mathbf{K}^+(x_1) = \mathbf{K}^-(x_1)$ ahead of the crack tip, and then using Eqs. (32), (36), (44) and (46), one can obtain the principal part of $\mathbf{T}^{(\mathrm{p})}(r)$ ahead of the right crack-tip. The result is

$$\mathbf{T}^{(\mathrm{p})}(r) = \mathbf{V}\langle\langle X_\alpha(x_1)\rangle\rangle\left[\sum_{j=1}^{4}\left\langle \mathbf{W}_j^{(1)} + \mathbf{I}\right\rangle \mathbf{M}^{(1)}\mathbf{I}_j\mathbf{q} + \sum_{j=1}^{4}\left\langle \mathbf{W}_j^{(2)} + \mathbf{I}\right\rangle \mathbf{M}^{(2)}\mathbf{I}_j\bar{\mathbf{q}}\right] \tag{47}$$

where $r$ means the distance from the crack-tip; $x_1 = a + r$; and

$$\mathbf{V} = \left(\mathbf{I} + \overline{\mathbf{H}}^{-1}\mathbf{H}\right)\mathbf{Q}\left\langle\left\langle \frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}}\right\rangle\right\rangle[\mathbf{I}+\mathbf{J}], \qquad \mathbf{J} = -\frac{1}{c_D}\mathbf{Q}^{-1}\mathbf{i}_4\mathbf{H}_4\mathbf{Q}\left\langle\left\langle \frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}}\right\rangle\right\rangle$$

Thus, the field intensity factor vector may be defined as

$$\begin{aligned}\mathbf{k}^{(\mathrm{p})} &= [k_{\mathrm{II}}, k_{\mathrm{I}}, k_{\mathrm{III}}, k_D]^{\mathrm{T}} \\ &= \lim_{r\to 0}\sqrt{2\pi r}\,\mathbf{V}\langle\langle r^{\mathrm{i}\varepsilon_\alpha}\rangle\rangle\langle\langle X_\alpha(x_1)\rangle\rangle\left[\sum_{j=1}^{4}\left\langle \mathbf{W}_j^{(1)} + \mathbf{I}\right\rangle \mathbf{M}^{(1)}\mathbf{I}_j\mathbf{q} + \sum_{j=1}^{4}\left\langle \mathbf{W}_j^{(2)} + \mathbf{I}\right\rangle \mathbf{M}^{(2)}\mathbf{I}_j\bar{\mathbf{q}}\right]\end{aligned} \tag{48}$$

Substituting Eq. (A.10) into Eq. (48) yields

$$\mathbf{k}^{(\mathrm{p})} = \sqrt{\pi/a}\,\mathbf{V}\left\langle\left\langle (2a)^{\mathrm{i}\varepsilon_\alpha}\right\rangle\right\rangle\left[\sum_{j=1}^{4}\left\langle \mathbf{W}_j^{(1)}(a) + \mathbf{I}\right\rangle \mathbf{M}^{(1)}\mathbf{I}_j\mathbf{q} + \sum_{j=1}^{4}\left\langle \mathbf{W}_j^{(2)}(a) + \mathbf{I}\right\rangle \mathbf{M}^{(2)}\mathbf{I}_j\bar{\mathbf{q}}\right] \tag{49}$$

If the crack is assumed to be impermeable, i.e. let $D_2(z) = 0$, this is equivalent to letting $\mathbf{i}_4$ be zero. Thus $\mathbf{J} = \mathbf{0}$. On the other hand, considering that

$$\begin{aligned}\mathbf{V} &= \mathbf{Q}\mathbf{Q}^{-1}\left(\mathbf{I} + \overline{\mathbf{H}}^{-1}\mathbf{H}\right)\mathbf{Q}\left\langle\left\langle \frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}}\right\rangle\right\rangle[\mathbf{I}+\mathbf{J}] = \mathbf{Q}\langle\langle 1+\mathrm{e}^{2\pi\varepsilon_\alpha}\rangle\rangle\left\langle\left\langle \frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}}\right\rangle\right\rangle[\mathbf{I}+\mathbf{J}] \\ &= \mathbf{Q}[\mathbf{I}+\mathbf{J}]\end{aligned} \tag{50}$$

we can readily give the solutions of crack-tip field and field intensity factor vector for an impermeable crack as

$$\mathbf{T}^{(\mathrm{im})}(r) = \mathbf{Q}\langle\langle X_\alpha(x_1)\rangle\rangle\left[\sum_{j=1}^{4}\left\langle \mathbf{W}_j^{(1)} + \mathbf{I}\right\rangle \mathbf{M}^{(1)}\mathbf{I}_j\mathbf{q} + \sum_{j=1}^{4}\left\langle \mathbf{W}_j^{(2)} + \mathbf{I}\right\rangle \mathbf{M}^{(2)}\mathbf{I}_j\bar{\mathbf{q}}\right] \tag{51}$$

$$\mathbf{k}^{(\mathrm{im})} = \sqrt{\pi/a}\,\mathbf{Q}\left\langle\left\langle (2a)^{\mathrm{i}\varepsilon_\alpha}\right\rangle\right\rangle\left[\sum_{j=1}^{4}\left\langle \mathbf{W}_j^{(1)}(a) + \mathbf{I}\right\rangle \mathbf{M}^{(1)}\mathbf{I}_j\mathbf{q} + \sum_{j=1}^{4}\left\langle \mathbf{W}_j^{(2)}(a) + \mathbf{I}\right\rangle \mathbf{M}^{(2)}\mathbf{I}_j\bar{\mathbf{q}}\right] \tag{52}$$

Eqs. (47) and (51) show that for two crack models, the singular structures of crack-tip filed are same, but the intensities of crack-tip filed are different. From Eqs. (49), (50) and (52), we obtain the relation between the field intensity factor vector in permeable crack model and that in impermeable crack model as:

$$\mathbf{k}^{(\mathrm{p})} = \left[\mathbf{I} + \mathbf{Q}\mathbf{J}\mathbf{Q}^{-1}\right]\mathbf{k}^{(\mathrm{im})} \tag{53}$$

For the case of a crack in a homogeneous piezoelectric material, noting that

$$\varepsilon_\alpha = 0, \qquad \mathbf{Q} = \mathbf{I}, \qquad \mathbf{M}^{(1)} = \mathbf{B}_1, \qquad \mathbf{M}^{(2)} = \overline{\mathbf{B}_1}, \qquad c_D = H_{44}/2, \qquad \mathbf{J} = -\frac{1}{H_{44}}\mathbf{i}_4\mathbf{H}_4 \tag{54}$$

one has from Eq. (53) that

$$\mathbf{k}^{(\mathrm{p})} = \left[\mathbf{I} - \frac{1}{H_{44}}\mathbf{i}_4\mathbf{H}_4\right]\mathbf{k}^{(\mathrm{im})} \tag{55}$$

Eqs. (53) and (55) show that if the field intensity factor vector of an impermeable crack is known, the corresponding value of a permeable crack is easily written out.

## 5. Several special examples

### 5.1. A permeable crack in a homogeneous medium subject to the uniform far-field loading

In this case, the field intensity factor vector of an impermeable crack is (Suo et al., 1992):

$$\mathbf{k}^{(\mathrm{im})} = \sqrt{\pi a}\left[\sigma_{21}^{\infty}, \sigma_{22}^{\infty}, \sigma_{23}^{\infty}, D_2^{\infty}\right]^{\mathrm{T}} \tag{56}$$

Inserting Eqs. (56) into (55), one can obtain the field intensity factor vector of a permeable crack as

$$k_j^{(\mathrm{p})} = \sqrt{\pi a}\sigma_{2j}^{\infty}, \quad (j = 1, 2, 3); \qquad k_4^{(\mathrm{p})} = -\frac{1}{H_{44}}\sum_{j=1}^{3} H_{4j} k_j^{(\mathrm{p})} \tag{57}$$

which are consistent with the results of Gao and Wang (2000).

### 5.2. A permeable crack in a homogeneous medium subject to an arbitrary concentration loading

Since $\mathbf{k}^{(\mathrm{im})}$ for this case is not known, we directly derive the $\mathbf{k}^{(\mathrm{p})}$ from Eq. (48). Substituting Eqs. (50) and (54) and (A.10) into Eq. (48) gives

$$\mathbf{k}^{(\mathrm{p})} = 2\sqrt{\pi/a}\,\mathrm{Re}\left[\mathbf{B}_1\left\langle\left\langle 1 - \sqrt{\frac{z_{\alpha 0} + a}{z_{\alpha 0} - a}}\right\rangle\right\rangle \mathbf{q}\right] - \frac{1}{H_{44}}\mathbf{i}_4\mathbf{H}_4 2\sqrt{\pi/a}\,\mathrm{Re}\left[\mathbf{B}_1\left\langle\left\langle 1 - \sqrt{\frac{z_{\alpha 0} + a}{z_{\alpha 0} - a}}\right\rangle\right\rangle \mathbf{q}\right] \tag{58}$$

Eq. (58) can be expanded into

$$k_j^{(\mathrm{p})} = 2\sqrt{\pi/a}\,\mathrm{Re}\sum_{\alpha=1}^{4} B_{1j\alpha} q_\alpha \left(1 - \sqrt{\frac{z_{\alpha 0} + a}{z_{\alpha 0} - a}}\right), \quad (j = 1\text{–}3) \tag{59}$$

$$\begin{aligned} k_4^{(\mathrm{p})} &= 2\sqrt{\pi/a}\,\mathrm{Re}\sum_{\alpha=1}^{4} B_{14\alpha} q_\alpha \left(1 - \sqrt{\frac{z_{\alpha 0} + a}{z_{\alpha 0} - a}}\right) - \frac{2\sqrt{\pi/a}}{H_{44}}\sum_{j=1}^{4} H_{4j}\,\mathrm{Re}\left(\sum_{\alpha=1}^{4} B_{1j\alpha} q_\alpha \left(1 - \sqrt{\frac{z_{\alpha 0} + a}{z_{\alpha 0} - a}}\right)\right) \\ &= -\frac{2\sqrt{\pi/a}}{H_{44}}\sum_{j=1}^{3} H_{4j}\,\mathrm{Re}\left(\sum_{\alpha=1}^{4} B_{1j\alpha} q_\alpha \left(1 - \sqrt{\frac{z_{\alpha 0} + a}{z_{\alpha 0} - a}}\right)\right) = -\frac{1}{H_{44}}\sum_{j=1}^{3} H_{4j} k_j \end{aligned} \tag{60}$$

Eqs. (59) and (60) are consistent to those of Gao and Fan (1998b), who started with an elliptic hole and obtained the corresponding solutions of a permeable crack.

### 5.2. An interface crack between two purely elastic media

In this case, on the crack faces one has $D_2(z) = 0$. Noting Eq. (50) and $\mathbf{J} = \mathbf{0}$ one can easily obtain the corresponding complex vector $\mathbf{R}(z)$ and the vector of field intensity factor. The results are

$$\begin{aligned} \mathbf{R}(z) = &-\left\langle\left\langle \frac{1}{1 + \mathrm{e}^{2\pi\varepsilon_\alpha}} \right\rangle\right\rangle \left[\mathbf{M}^{(1)}\mathbf{G}_{10}(z) + \mathbf{M}^{(2)}\overline{\mathbf{G}_{10}}(z)\right] + \left\langle\left\langle \frac{1}{1 + \mathrm{e}^{2\pi\varepsilon_\alpha}} \right\rangle\right\rangle \langle\langle X_\alpha(z)\rangle\rangle \\ &\times \left[\sum_{j=1}^{3}\left\langle \mathbf{W}_j^{(1)} + \mathbf{I}\right\rangle \mathbf{M}^{(1)}\mathbf{I}_j\mathbf{q} + \sum_{j=1}^{3}\left\langle \mathbf{W}_j^{(2)} + \mathbf{I}\right\rangle \mathbf{M}^{(2)}\mathbf{I}_j\overline{\mathbf{q}}\right] \end{aligned} \tag{61}$$

$$\mathbf{k} = \sqrt{\pi/a}\mathbf{Q}\left\langle\left\langle (2a)^{i\varepsilon_\alpha}\right\rangle\right\rangle\left[\sum_{j=1}^{3}\left\langle \mathbf{W}_j^{(1)}(a) + \mathbf{I}\right\rangle \mathbf{M}^{(1)}\mathbf{I}_j\mathbf{q} + \sum_{j=1}^{3}\left\langle \mathbf{W}_j^{(2)}(a) + \mathbf{I}\right\rangle \mathbf{M}^{(2)}\mathbf{I}_j\bar{\mathbf{q}}\right] \tag{62}$$

It should be noted in this case that all the matrices and vectors in Eqs. (61) and (62) degenerate into three degree ones.

### 5.3. The case of a bimaterial of piezoelectric materials without crack

In this case, the perfect connection condition along the $x_1$ axis requires from Eq. (22) that

$$\mathbf{H}[\mathbf{K}^+(x_1) - \mathbf{K}^-(x_1)] = \mathbf{0}, \quad -\infty < x_1 < \infty \tag{63}$$

Using Eq. (24), we have from Eq. (63) that

$$\mathbf{K}(z) = 0 \tag{64}$$

Substituting Eq. (64) into Eq. (23) results in

$$\begin{aligned} \mathbf{B}_1\mathbf{F}_{10}(z) &= \mathbf{H}^{-1}\left(\overline{\mathbf{Y}}_1 - \overline{\mathbf{Y}}_2\right)\overline{\mathbf{B}_1}\,\overline{\mathbf{G}_{10}}(z), \quad z \in S_1 \\ \mathbf{B}_2\mathbf{F}_{20}(z) &= \overline{\mathbf{H}}^{-1}\left(\overline{\mathbf{Y}}_1 + \mathbf{Y}_1\right)\mathbf{B}_1\mathbf{G}_{10}(z), \quad z \in S_2 \end{aligned} \tag{65}$$

If the upper space and lower space consist of an identical material, Eqs. (12) and (65) lead to

$$\mathbf{F}_1(z) = \mathbf{F}_2(z) = \mathbf{G}_{10}(z) \tag{66}$$

which is obvious.

## 6. Conclusions

The generalized 2D problem of a permeable interfacial crack between two dissimilar piezoelectric media, one of which is subjected to a line load at an arbitrary point, is derived by using the Stroh's formalism combined with the technique of analytical continuations. The fundamental solutions for the Green's function and the field intensity factor are presented in explicit closed-form, and thus they are very useful for solving some complicated problems in engineering. It can be found that despite the mathematical complexities inherent to this problem, the present analysis is very straightforward and explicit. This is attributable to a combination of the Stroh's formalism with the Muskhelishvili's theory. This combination makes the Stroh's formalism more powerful and elegant in analyzing the generalized 2D problems of anisotropic media, meantime it gives full play to the well-established Muskhelishvili's theory.

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## Appendix A

Eq. (37) can be expanded into

$$R_\alpha^+(x_1) - g_\alpha R_\alpha^-(x_1) = f_\alpha(x_1), (\alpha = 1 - 4) \tag{A.1}$$

where

$$g_\alpha = \mathrm{e}^{-2\pi \mathrm{i}\delta_\alpha} = -\mathrm{e}^{2\pi\varepsilon_\alpha} \tag{A.2}$$

$$f_\alpha(x_1) = M_\alpha^{(0)} D_2(x_1) - \sum_{j=1}^{4} M_{\alpha j}^{(1)} q_j \left(x_1 - z_{j0}\right)^{-1} - \sum_{j=1}^{4} M_{\alpha j}^{(2)} \overline{q_j} \left(x_1 - \overline{z_{j0}}\right)^{-1} \tag{A.3}$$

According to Muskhelishvili (1975, Chapter 6, 110), the general solution of (A.1) is

$$R_\alpha(z) = X_\alpha(z) I_\alpha(z) + X_\alpha(z) P_\alpha(x_1) \tag{A.4}$$

where

$$I_\alpha(z) = \frac{1}{2\pi \mathrm{i}} \int_{L_c} \frac{f_\alpha(x_1)dx_1}{X_\alpha^+(x_1)(x_1 - z)} \tag{A.5}$$

$$P_\alpha(z) = c_{\alpha 1}^{(1)} z + c_{\alpha 1}^{(0)} \tag{A.6}$$

$$X_\alpha(z) = (z + a)^{-\gamma_\alpha} (z - a)^{\gamma_\alpha - 1} \tag{A.7}$$

Note

$$\gamma_\alpha = \frac{\ln g_\alpha}{2\pi \mathrm{i}} = -\delta_\alpha = \frac{1}{2} - \mathrm{i}\varepsilon_\alpha \tag{A.8}$$

$$\gamma_\alpha - 1 = -\frac{1}{2} - \mathrm{i}\varepsilon_\alpha \tag{A.9}$$

Then, (A.7) can be rewritten as

$$X_\alpha(z) = \frac{1}{\sqrt{z^2 - a^2}} \left(\frac{z + a}{z - a}\right)^{\mathrm{i}\varepsilon_\alpha} \tag{A.10}$$

On the other hand, (A.5) can be reduced to

$$I_\alpha(z) = \frac{1}{1 - g_\alpha} \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{f_\alpha(t)dt}{X_\alpha^+(t)(t - z)} \tag{A.11}$$

where $t$ is any point on $\Gamma$, which stands for a clockwise closed-contour closing in on the crack.

Using the Cauchy integration principle (Muskhelishvili, 1975, Chapter 4, 70), (A.11) results in

$$I_\alpha(z) = \frac{1}{1 - g_\alpha} \left[ \frac{f_\alpha(z)}{X_\alpha(z)} - \sum_{j=1}^{3} (I_j^0 + I_j^\infty) \right] \tag{A.12}$$

where $I_j^0$ and $I_j^\infty$ are the principle parts of the $j$-th term of $f_\alpha(z)/X_\alpha(z)$, respectively, at $z_0$ and at infinity. Considering that $D_2(\infty) = 0$ and the nature of $D_2(z_0)$ from (12), (1) and (4), $I_j^0$ and $I_j^\infty$ can be calculated, respectively, by

$$I_1^0 = \beta_{\alpha 2}^{(1)} z + \beta_{\alpha 2}^{(0)}, \quad I_2^0 = -\sum_{j=1}^{4} M_{\alpha j}^{(1)} q_j \frac{X_\alpha^{-1}\left(z_{j0}\right)}{z - z_{j0}}, \quad I_3^0 = -\sum_{j=1}^{4} M_{\alpha j}^{(2)} \overline{q_j} \frac{X_\alpha^{-1}\left(\overline{z_{j0}}\right)}{z - \overline{z_{j0}}} \tag{A.13}$$

$$I_1^\infty = 0, \quad I_2^\infty = -\sum_{j=1}^{4} M_{\alpha j}^{(1)} q_j, \quad I_3^\infty = -\sum_{j=1}^{4} M_{\alpha j}^{(2)} \overline{q_j} \tag{A.14}$$

where $c_{\alpha 2}^{(1)}$ and $c_{\alpha 2}^{(0)}$ are unknown constants.

Substituting (A.6) and (A.12) together with (A.3), (A.13) and (A.14) into (A.4), we finally obtain

$$R_\alpha(z) = \frac{1}{1+e^{2\pi\varepsilon_\alpha}}\left[M_\alpha^{(0)}D_2(z) - \sum_{j=1}^{4} M_{\alpha j}^{(1)} q_j (z - z_{j0})^{-1} - \sum_{j=1}^{4} M_{\alpha j}^{(2)} \overline{q_j}(z - \overline{z_{j0}})^{-1}\right] + \frac{X_\alpha(z)}{1+e^{2\pi\varepsilon_\alpha}}\left[\sum_{j=1}^{4} M_{\alpha j}^{(1)} q_j\left(1 + \frac{X_\alpha^{-1}(z_{j0})}{z - z_{j0}}\right) + \sum_{j=1}^{4} M_{\alpha j}^{(2)} \overline{q_j}\left(1 + \frac{X_\alpha^{-1}(\overline{z_{j0}})}{z - \overline{z_{j0}}}\right)\right] + \frac{1}{1+e^{2\pi\varepsilon_\alpha}} X_\alpha(z)\left[c_\alpha^{(1)} z + c_\alpha^{(0)}\right] \tag{A.15}$$

where $c_\alpha^{(1)}$ and $c_\alpha^{(0)}$ are to new constants to be found.

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